MEM6804 Modeling and Simulation for Logistics & Supply Chain 物流与供应链建模与仿真

Theory | Analysis

Lecture 2: Elements of Probability and Statistics

SHEN Haihui 沈海辉

Sino-US Global Logistics Institute
Shanghai Jiao Tong University

shenhaihui.github.io/teaching/mem6804f
shenhaihui@sjtu.edu.cn

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上海交通大学 SHANGHAI JIAO TONG UNIVERSITY | 董浩云航运与物流研究院 CY TUNG Institute of Maritime and Logistics
中美物流研究院（工程系统管理研究院） Sino-US Global Logistics Institute (Institute of Industrial & System Engineering)

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  3. Countably additive: If $A_i \in \mathcal{F}$, $i = 1, 2, \ldots$, is a **countable** sequence of **disjoint** sets, then $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$.

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- Example 1: Flip a fair coin.
  - $\Omega = \{\text{H (head)}, \text{T (tail)}\}$;
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  - A more practical and interesting $\mathcal{F}$ is the one that contains all intervals (no matter open or closed) on $[0, 1]$.

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- Events A and B are independent $\iff \mathbb{P}(A|B) = \mathbb{P}(A)$.

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- Clearly, mutual independence implies pairwise independence, but not vice versa!

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- Remark: For event A, if $\mathbb{P}(A) = 1$, then we say A happens **almost surely** (a.s.).

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    - Usually, we will simply denote $X(\omega)$ as x when ω is not explicitly shown.
    - A popular convention is to denote the RVs by upper-case letters (e.g., X and Y) and their realizations by lower-case letters (e.g., x and y).

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  - Under $(\Omega, \mathcal{F}, \mathbb{P})$, let $X(\omega) = \omega$ for $\omega \in [0, 1]$.

- The **cumulative distribution function** (CDF) of a RV X, denoted by $F:\mathbb{R}\to[0,1]$, is defined by

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  - $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = 1$;
  - $F(x)$ is nondecreasing in x;
  - $F(x)$ is right-continuous, that is, for any $x_0 \in \mathbb{R}$,

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- It is easy to see that $F(x) = \sum_{y \in (-\infty, x]} p(y)$.

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- The **joint** CDF of RVs X and Y, denoted by $F : \mathbb{R}\times\mathbb{R} \to [0,1]$, is defined by

$$\begin{aligned} F(x,y) &:= \mathbb{P}(X \le x, Y \le y) \\ &= \mathbb{P}(\{\omega : X(\omega) \le x\} \cap \{\omega : Y(\omega) \le y\}), \ \forall x, y \in \mathbb{R}. \end{aligned}$$

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- Observe that $\frac{\partial^2 F(x,y)}{\partial x \partial y} = f(x,y)$.

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- For Y, its marginal CDF, and pmf or pdf, can be determined similarly.

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2. Then, $f(x|y) = \frac{\partial}{\partial x} F(x|Y=y) = \frac{\frac{\partial}{\partial x}\int_{-\infty}^{x} f(t,y)\mathrm{d}t}{f_Y(y)} = \frac{f(x,y)}{f_Y(y)}$.

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- RVs $X_1, \dots, X_n$ are pairwise independent if for any $i \neq j$, $X_i \perp X_j$.

- The **expectation**, or **expected value**, or **mean**, of a RV X is defined as

$$\mathbb{E}[X] := \int_\Omega X(\omega) \mathrm{d}\,\mathbb{P}(\omega),$$

provided that $\int_\Omega |X(\omega)| \mathrm{d}\,\mathbb{P}(\omega) < \infty$ or $X \geq 0$ a.s., where the integral is the Lebesgue integral, rather than the Riemann integral.

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- In general, $X \perp Y \;\underset{\not\Longleftarrow}{\Longrightarrow}\; \rho(X,Y) = 0 \iff \mathrm{Cov}(X,Y) = 0$.
- If $(X,Y)^\intercal$ follows a bivariate normal distribution,[†] then $X \perp Y \iff \rho(X,Y) = 0$.

[†] **CAUTION:** It means MORE than that X and Y both follow a normal distribution! More details latter.

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- The conditional expectation of X given $Y = y$ is

$$\mathbb{E}[X|y] := \begin{cases} \sum_{x\in\mathbb{R}} xp(x|y), & \text{if } X \text{ is discrete,} \\ \int_{-\infty}^{+\infty} xf(x|y)\mathrm{d}x, & \text{if } X \text{ is continuous.} \end{cases}$$

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- If $X \perp Y$, then $\mathbb{E}[X|y] = \mathbb{E}[X|Y] = \mathbb{E}[X]$, and $\mathrm{Var}(X|y) = \mathrm{Var}(X|Y) = \mathrm{Var}(X)$.

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  - Given $X_1 + X_2 = n$, $X_1 \sim \mathrm{B}(n, \lambda_1/(\lambda_1 + \lambda_2))$.

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- If $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$ are independent, then $\min\{X_1, X_2\} \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$.

- If $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$ are independent, then $\min\{X_1, X_2\} \sim \mathrm{Exp}(\lambda_1 + \lambda_2)$.
- If $X \sim \mathrm{Exp}(\lambda)$, then for $\alpha > 0$, $Y := X^{1/\alpha} \sim \mathrm{Weibull}(\alpha, \beta)$ in shape & scale parametrization with $\beta = (1/\lambda)^{1/\alpha}$, whose pdf is
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  - $\alpha = 1, \beta > 1 \implies$ strictly decreasing
  - $\alpha < 1, \beta < 1 \implies$ U-shaped
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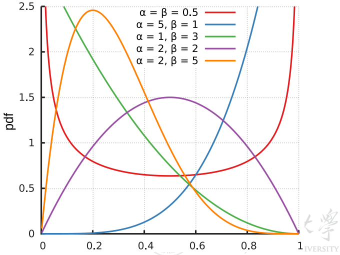

- $X \sim$ Student's t distribution with p degrees of freedom, denoted as t_p, where p is an integer, if its pdf is given by

$$f(x) = \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1 + x^2/p)^{(p+1)/2}}, \quad x \in \mathbb{R}.$$

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Proof. Let $Y := Z^2$. For $y \in [0, \infty)$,

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The proof is completed by showing that $\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}}e^{-t}\mathrm{d}t = \sqrt{\pi}$, which can be seen if we convert to polar coordinates. $\blacksquare$

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Let $Y := \sqrt{V/p}$. For $y \in [0, \infty)$,

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- If $Z \sim \mathcal{N}(0,1)$ and $V \sim \chi_p^2$ are independent, then $\frac{Z}{\sqrt{V/p}} \sim t_p$.

Proof. Since $V \sim \chi_p^2$, by definition, its pdf is

$$f_V(v) = \frac{(\frac{1}{2})^{\frac{p}{2}}}{\Gamma(\frac{p}{2})} v^{\frac{p}{2}-1} e^{-\frac{1}{2}v}, \quad v \in [0,\infty).$$

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Let $T := \frac{Z}{\sqrt{V/p}} = \frac{Z}{Y}$. For $t \in \mathbb{R}$,

$$\mathbb{P}(T \le t) = \mathbb{P}\left(\frac{Z}{Y} \le t\right) = \mathbb{P}(Z \le tY) = \int_0^{\infty} \mathbb{P}(Z \le ty) f_Y(y)\mathrm{d}y. \quad \text{(Why?)}$$

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Then,

$$f_T(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(T \le t) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(Z \le ty) f_Y(y)\mathrm{d}y.$$

Proof. (Cont'd) Note that $\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{P}(Z \le ty) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{ty}\phi(z)\mathrm{d}z = y\phi(ty)$.

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&= \int_0^\infty 2py^2 \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2y^2}{2}} \cdot \frac{\left(\frac{1}{2}\right)^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}\right)}(py^2)^{\frac{p}{2}-1}e^{-\frac{1}{2}py^2}\mathrm{d}y
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Let $x := y^2$. Then, integration by substitution shows that

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$$\begin{aligned} f_T(t) &= \frac{1}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} 2^{\frac{1-p}{2}} p^{\frac{p+1}{2}} \cdot \frac{1}{2} \frac{\Gamma(\frac{p+1}{2})}{(1/2)^{(p+1)/2}(t^2+p)^{(p+1)/2}} \\ &= \frac{\Gamma(\frac{p+1}{2})}{\Gamma(\frac{p}{2})} \frac{1}{(p\pi)^{1/2}} \frac{1}{(1+t^2/p)^{(p+1)/2}}. \end{aligned}$$

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- If $\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is k dimensional, then
  - $\boldsymbol{Z} := \boldsymbol{A}^{-1}(\boldsymbol{X} - \boldsymbol{\mu}) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$, where $\boldsymbol{A}$ satisfies $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}^\intercal$ (Cholesky decomposition), $\boldsymbol{0} \in \mathbb{R}^k$, and $\boldsymbol{I} \in \mathbb{R}^{k \times k}$ denotes the identity matrix.

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† The multivariate normal distribution will be degenerate if $\boldsymbol{B}$ does not have full row rank ($\boldsymbol{B}$ 不行满秩).

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- Bivariate normal distribution: $(X_1, X_2)^\intercal \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} = (\mu_1, \mu_2)^\intercal$, and

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  and the joint pdf is

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}.$$

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- To see $\rho = 0 \Longrightarrow X_1 \perp X_2$, let $\rho = 0$, and note

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]} \\ &= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}} \times \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}} = f_{X_1}(x_1) f_{X_2}(x_2). \end{aligned}$$

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  Since $\boldsymbol{B}$ has full row rank, $\boldsymbol{Y} \sim \mathcal{N}(\boldsymbol{B}\boldsymbol{\mu}, \boldsymbol{B}\boldsymbol{\Sigma}\boldsymbol{B}^\intercal)$, which is non-degenerate.

- If $(X_1, X_2)^\intercal \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $X_i \sim \mathcal{N}(\mu_i, \sigma^2)$, $i = 1, 2$, then $X_1 + X_2 \perp X_1 - X_2$.

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$$\begin{aligned} \mathrm{Cov}(X_1 + X_2, X_1 - X_2) &= \mathrm{Cov}(X_1, X_1) - \mathrm{Cov}(X_2, X_2) \\ &= \sigma^2 - \sigma^2 = 0. \end{aligned}$$

■

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- There are many other relationships among various probability distributions.
  - See, for example, Song (2005);
  - Or, Leemis & McQueston (2008) and their online interactive graph `http://www.math.wm.edu/~leemis/chart/UDR/UDR.html`

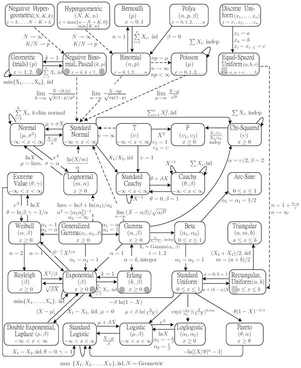

Figure: Relationships Among 35 Distributions (*from* Song (2005))

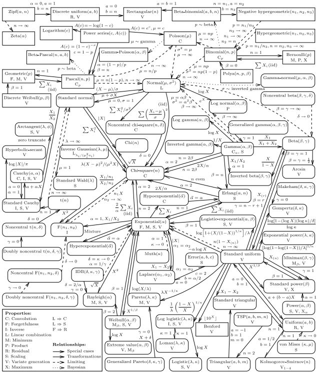

Figure: Relationships Among 76 Distributions (*from* Leemis & McQueston (2008))

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Markov's Inequality

Let X be a RV. If $\mathbb{P}(X \geq 0) = 1$ and $\mathbb{P}(X = 0) < 1$, then, for any $r > 0$,

$$\mathbb{P}(X \geq r) \leq \frac{\mathbb{E}[X]}{r},$$

with equality if and only if

$$X = \begin{cases} r, & \text{with probability } p, \\ 0, & \text{with probability } 1 - p. \end{cases}$$

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- Markov's Inequality has many variations, which are usually called Chebyshev's Inequality.

Chebyshev's Inequality

Let X be a RV and $g(x)$ be a nonnegative function. Then, for any $r > 0$,

$$\mathbb{P}(g(X) \geq r) \leq \frac{\mathbb{E}[g(X)]}{r}.$$

Chebyshev's Inequality

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Chebyshev's Inequality

Let X be a RV. Then, for any $r, p > 0$,

$$\mathbb{P}(|X| \geq r) \leq \frac{\mathbb{E}[|X|^p]}{r^p},$$

$$\mathbb{P}(|X - \mu| \geq r) \leq \frac{\sigma^2}{r^2},$$

where $\mu := \mathbb{E}[X]$, and $\sigma^2 := \mathrm{Var}(X)$.

- Chebyshev's Inequality is typically very conservative.

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- If $Z \sim \mathcal{N}(0,1)$, a tighter bound is available: For any $t > 0$,

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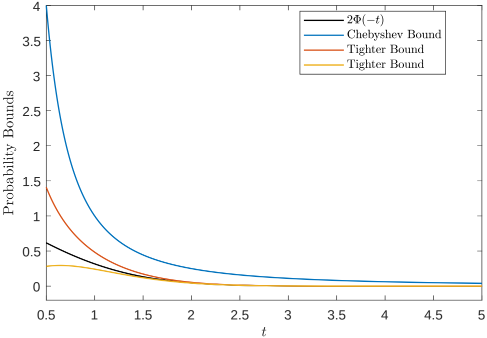

- A function $g(x)$ is **convex** if

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y),$$

for all x and y, and $\lambda \in (0,1)$.

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Jensen's Inequality

Let X be a RV. If $g(x)$ is a convex function, then

$$\mathbb{E}[g(X)] \geq g(\mathbb{E}[X]),$$

with equality if and only if $g(x)$ is a linear function on some set A such that $\mathbb{P}(X \in A) = 1$.

Hölder's Inequality

Let X and Y be any two RVs, and let p and q be any two positive numbers (necessarily greater than 1) satisfying

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Then,

$$|\mathbb{E}[XY]| \leq \mathbb{E}[|XY|] \leq \{\mathbb{E}[|X|^p]\}^{1/p}\{\mathbb{E}[|Y|^q]\}^{1/q}.$$

Cauchy-Schwarz Inequality ($p = q = 2$)

Let X and Y be any two RVs, then

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Liapounov's Inequality ($Y \equiv 1$)

Let X be a RV, then for any $s > r > 1$,

$$\{\mathbb{E}[|X|^r]\}^{1/r} \le \{\mathbb{E}[|X|^s]\}^{1/s}.$$

Minkowski's Inequality

Let X and Y be any two RVs. Then, for $p \geq 1$,

$$\{\mathbb{E}[|X+Y|^p]\}^{1/p} \leq \{\mathbb{E}[|X|^p]\}^{1/p} + \{\mathbb{E}[|Y|^p]\}^{1/p}.$$

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- **Remark**: The preceding Hölder's Inequality (including its special cases) and Minkowski's Inequality also apply to numerical sums where there is no explicit reference to an expectation.

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- **Convergence in L^r Norm** ($r \in [1, \infty)$), $X_n \xrightarrow{L^r} X$:

$$\lim_{n\to\infty} \mathbb{E}(|X_n - X|^r) = 0,$$

given $\mathbb{E}[|X_n|^r] < \infty$ for any $n \geq 1$ and $\mathbb{E}[|X|^r] < \infty$.

- Simple relationships:

$$
\begin{array}{ccccc}
X_n \xrightarrow{a.s.} X & \Longrightarrow & X_n \xrightarrow{p} X & \Longrightarrow & X_n \Rightarrow X \\
 & & \Uparrow & & \\
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- $X_n \xrightarrow{p} X \quad \Longleftrightarrow$ For every subsequence $X_n(m)$ there is a further subsequence $X_n(m_k)$ such that $X_n(m_k) \xrightarrow{a.s.} X$.

- Question: If $X_n \Rightarrow X$ or $X_n \xrightarrow{p} X$ or $X_n \xrightarrow{a.s.} X$, does it imply $\mathbb{E}[X_n] \to \mathbb{E}[X]$?

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Monotone Convergence Theorem (MCT)

Suppose $X_n \xrightarrow{a.s.} X$, and $0 \le X_1 \le X_2 \le \cdots$ a.s.. Then $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

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Fatou's Lemma

Suppose $X_n \ge Y$ a.s. for all n where $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[\liminf_{n\to\infty} X_n] \le \liminf_{n\to\infty} \mathbb{E}[X_n]$. In particular, if $X_n \ge 0$ a.s. for all n, then the result holds.

Dominated Convergence Theorem (DCT)

Suppose $X_n \xrightarrow{a.s.} X$, $|X_n| \leq Y$ a.s. for all n, and $\mathbb{E}[|Y|] < \infty$. Then $\mathbb{E}[X_n] \to \mathbb{E}[X]$.

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- An **even more general** result:
  Suppose $X_n \xrightarrow{p} X$, $|X_n| \leq Y$ a.s. for all n, and $\mathbb{E}[|Y|^r] < \infty$ with $r \geq 1$. Then, $\mathbb{E}[|X_n|^r] < \infty$, $\mathbb{E}[|X|^r] < \infty$, and $X_n \xrightarrow{L^r} X$.

- $X = Y$ a.s., if *any one* of the following holds:
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- None of the above are true for convergence in distribution.
- If $X_n \Rightarrow X$ and $Y_n \Rightarrow$ **constant** c, then $(X_n, Y_n)^\intercal \Rightarrow (X, c)^\intercal$.
  $\Longrightarrow aX_n + bY_n \Rightarrow aX + bc$; $X_nY_n \Rightarrow cX$. (Due to CMT; also known as Slutsky's theorem)

Continuous Mapping Theorem (CMT)

Consider a sequence of RVs $\{X_n : n \geq 1\}$ and another RV X. Suppose g is a function that has the set of discontinuity points D such that $\mathbb{P}(X \in D) = 0$. Then,

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\begin{aligned}
X_n \xrightarrow{a.s.} X \quad &\Longrightarrow \quad g(X_n) \xrightarrow{a.s.} g(X); \\
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$$\begin{aligned} X_n \xrightarrow{a.s.} X \quad &\Longrightarrow \quad g(X_n) \xrightarrow{a.s.} g(X); \\ X_n \xrightarrow{p} X \quad &\Longrightarrow \quad g(X_n) \xrightarrow{p} g(X); \\ X_n \Rightarrow X \quad &\Longrightarrow \quad g(X_n) \Rightarrow g(X). \end{aligned}$$

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- CMT also holds for **random vectors**.
- Caution: For convergence in L^r norm, stronger assumption of g than continuity is required to ensure $g(X_n) \xrightarrow{L^r} g(X)$.

Properties of a Random Sample

- Let $X_1, \ldots, X_n$ be a **random sample** from a distribution with mean μ and variance σ^2, i.e., $X_1, \ldots, X_n$ are iid, and $\mathbb{E}[X_i] = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$, $i = 1, \ldots, n$.

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Weak Law of Large Numbers (WLLN)

Suppose $X_1, \dots, X_n$ are iid with mean μ and variance $\sigma^2 < \infty$.[†] Then, $\bar{X}_n \xrightarrow{p} \mu$.

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Strong Law of Large Numbers (SLLN)

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- Note that for **normal** distribution, $\frac{\bar{X}_n-\mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0,1)$, regardless of the value of n.
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Central Limit Theorem (CLT)

Suppose $X_1, \ldots, X_n$ are iid with mean μ and variance $\sigma^2 \in (0, \infty)$. Then,

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \Rightarrow \mathcal{N}(0,1).$$